PC 12 **SECTION 6.3: PROVING IDENTITIES**

INVESTIGATE THE EQUIVALENCE OF TWO TRIGONOMETRIC EXPRESSIONS:

- Many formulas in science (especially Physics) contain trigonometric functions
- Identities can reduce the time it takes to work with formulas with trigonometric functions
  ex. torque (τ): $\tau = rF\ sin\theta$ work (W): $W = F\delta r\ cos\theta$ magnetic forces (F_B): $F_B = qvB\ sin\theta$

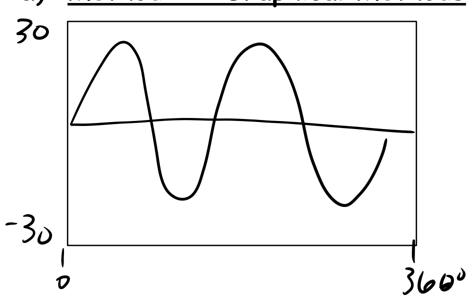
A model rocket that is launched with an angle of elevation θ is modeled by each equation:

$d = \frac{(v_0)^2 sin2\theta}{g}$ OR $d = \frac{2(v_0)^2}{g}(tan\theta - tan\theta sin^2\theta)$

d: horizontal distance
θ: angle of elevation
g: force of gravity
v_0: initial velocity

$y_1 = \frac{(14)^2 sin2\theta}{9.8}$ $y_2 = \frac{2(14)^2}{9.8}(tan\theta - tan\theta sin^2\theta)$

1. Verify the two equations are equivalent when $v_0 = 14\ m/s$ and $g = 9.8\ m/s^2$
   a) **Method 1: Graphical Methods** b) **Method 2: Numerical Methods** ex. $\theta = 45°$



$d = \frac{(14)^2 sin(2\cdot 45°)}{9.8}$ $d = \frac{2(14)^2}{9.8}(tan 45° - tan 45° sin^2 45°)$

$d = \frac{(14)^2 sin\ 90°}{9.8}$ $= 1$ $= \frac{2(14)^2}{9.8}\left(1 - 1\left(\frac{1}{\sqrt{2}}\right)^2\right)$

$d = \frac{(14)^2}{9.8}$ $= \frac{2(14)^2}{9.8}\left(\frac{1}{2}\right)$

$d = 20\ m$ $= 20\ m$

Which parts are common to both formulas?

$\frac{(v_0)^2}{g}$

2. Write an identity with the parts of the formulas that are not common.

$sin2\theta = 2(tan\theta - tan\theta sin^2\theta)$

3. Use your knowledge of identities to rewrite each side and show that they are equivalent.

$2sin\theta cos\theta$	$2tan\theta(1 - sin^2\theta)$
	$2tan\theta(cos^2\theta)$
	$2\frac{sin\theta}{\cancel{cos\theta}}(cos^{\cancel{2}}\theta) = 2sin\theta cos\theta$

4. Why do numerical and graphical verification fail to prove that an identity is true?

numerical - because it only shows that it works for the values you choose

graph - you can only tell that they're equal for the window you are looking at

VERIFY VERSUS PROVE THAT AN EQUATION IS AN IDENTITY

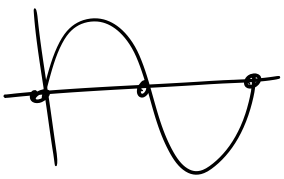
- **Trigonometric Identity**: a trigonometric equation that is true for all permissible values of the variable in the expressions on both sides of the equation.
  ➔ We can use identities to simplify expressions OR to prove other identities

  ① We can **VERIFY** identities
  - numerically by substituting a value in for the variable. Result: the LHS = RHS
  - graphically by graphing the LHS→Y_1 and RHS → Y_2. Result: identical graphs

  ② We can **PROVE** identities by
  - algebraically Result: the LHS = RHS for all values of x

1) complex fractions
2) factoring
3) conjugates
4) common denominators

EX. 1: a) Verify that $1 - sin^2x = sinx\ cosx\ cotx$ for some values of x.
Determine the non-permissible value(s) for x. Work in degrees.



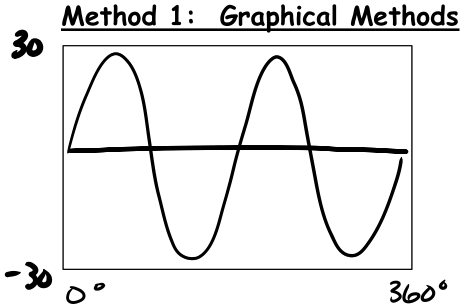
$\cot x = \frac{\cos x}{\sin x}$ $\quad \sin x \neq 0$

$x \neq \pi n, n \in I$

Use $x = 30°$

$\pi/6$.

Method 1: Graphical Methods



D: $[-360°, 360°]$ R: $[-3, 3]$

$Y_1 = 1 - sin^2x$
$Y_2 = sinx\ cosx\ cotx$

Method 2: Numerical Methods

$1 - sin^2x = sinx\ cosx\ cotx$

$1 - \sin^2(\pi/6) = \sin(\pi/6)\cos(\pi/6)\cot(\pi/6)$

$1 - \left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{1}\right)$

$1 - \frac{1}{4} = \frac{3}{4}$

$\frac{3}{4} = \frac{3}{4}$

b) Prove for all permissible values :

$1 - sin^2x = sinx\ cosx\ cotx$

$\cos^2 x$	$\cancel{\sin x} \cos x \frac{\cos x}{\cancel{\sin x}}$
	$\cos^2 x$

MATHEMATICAL TECHNIQUES USED FOR PROVING IDENTITIES

- In addition to the basic identities we have been using, we will learn a few more techniques **(complex fractions, factoring, conjugates,common denominators)** to prove identities.

A. COMPLEX FRACTIONS

- Simplify the following: $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{5}{6}+1}$ $\dfrac{\frac{3}{6}+\frac{2}{6}}{\frac{5}{6}+\frac{6}{6}} = \dfrac{\frac{5}{6}}{\frac{11}{6}} = \frac{5}{\cancel{6}}\cdot\frac{\cancel{6}}{11} = \frac{5}{11}$

EX. 2: Use the technique you learned above to prove the following identities:

a) $$\frac{1+cos\theta}{1+sec\theta} = cos\theta$$

$$\frac{1+\cos\theta}{1+\frac{1}{\cos\theta}}$$

$$\frac{1+\cos\theta}{\frac{\cos\theta}{\cos\theta}+\frac{1}{\cos\theta}}$$

$$\frac{1+\cos\theta}{\frac{\cos\theta+1}{\cos\theta}}$$

$$\cancel{1+\cos\theta}\cdot\frac{\cos\theta}{\cancel{\cos\theta+1}} \qquad \cos\theta = \cos\theta$$

b) $$\frac{1+tan\theta}{1+cot\theta} = \frac{1-tan\theta}{cot\theta-1}$$

$$\frac{\boxed{\frac{\cos\theta}{\cos\theta}}1+\frac{\sin\theta}{\cos\theta}}{\frac{\sin\theta}{\sin\theta}1+\frac{\cos\theta}{\sin\theta}} \quad\Bigg|\quad \frac{1-\frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\sin\theta}-1}$$

$$\frac{\frac{\cos\theta+\sin\theta}{\cos\theta}}{\frac{\sin\theta+\cos\theta}{\sin\theta}} \quad\Bigg|\quad \frac{\frac{\cos\theta-\sin\theta}{\cos\theta}}{\frac{\cos\theta-\sin\theta}{\sin\theta}}$$

$$\frac{\cancel{\cos\theta+\sin\theta}}{\cos\theta}\cdot\frac{\sin\theta}{\cancel{\sin\theta+\cos\theta}} \quad\Bigg|\quad \frac{\cancel{\cos\theta-\sin\theta}}{\cos\theta}\cdot\frac{\sin\theta}{\cancel{\cos\theta-\sin\theta}}$$

$$\frac{\sin\theta}{\cos\theta} \quad\Bigg|\quad \frac{\sin\theta}{\cos\theta}$$

c) $\frac{sin\theta+tan\theta}{1+cos\theta} = sin\theta sec\theta$

$$\frac{\sin\theta + \frac{\sin\theta}{\cos\theta}}{1+\cos\theta}$$

$$\frac{\frac{\sin\theta\cos\theta+\sin\theta}{\cos\theta}}{1+\cos\theta}$$

$$\frac{\sin(\cancel{\cos\theta+1})}{\cos\theta} \cdot \frac{1}{\cancel{\cos\theta+1}}$$

$$\frac{\sin\theta}{\cos\theta}$$

$$\sin\theta\left(\frac{1}{\cos\theta}\right)$$

$$\frac{\sin\theta}{\cos\theta}$$

C. FACTORING (GCF, difference of squares, trinomial factoring, decomposition...)

- Factor the difference of squares $a^2 - b^2 =$ $(a+b)(a-b)$

EX. 3: Use the technique you learned above to prove the following identity:

$$sin^4\theta - cos^4\theta = sin^2\theta - cos^2\theta$$

$$(\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta)$$

$$(1)(\sin^2\theta - \cos^2\theta$$

$$\sin^2\theta - \cos^2\theta$$

EX. 4: Prove for all permissible values

$$\frac{sin2x - cosx}{4sin^2x - 1} = \frac{sin^2x\ cosx + cos^3x}{2sinx + 1}$$

$\frac{2sinxcosx - cosx}{(2sinx+1)(2sinx-1)}$	$\frac{cosx(\underbrace{sin^2x + cos^2x}_{=1})}{2sinx+1}$
$\frac{cosx\cancel{(2sinx-1)}}{(2sinx+1)\cancel{(2sinx-1)}}$	$\frac{cosx}{2sinx+1}$
$\frac{cosx}{2sinx+1}$	

D. CONJUGATE TRICK

The conjugate of the denominator is: $\mathbf{1 - \sqrt{3}}$

- Rationalize the denominator: $\frac{2}{1+\sqrt{3}} \cdot \frac{(1-\sqrt{3})}{(1-\sqrt{3})} = \frac{2-2\sqrt{3}}{1-\sqrt{3}+\sqrt{3}-\sqrt{9}} = \frac{2-2\sqrt{3}}{1-3} = \frac{2-2\sqrt{3}}{-2}$

$= \frac{1-\sqrt{3}}{-1}$

$= -(1-\sqrt{3})$

$= -1+\sqrt{3}$

EX. 5: Use the technique you learned above to prove the following identity:

$$\boxed{\frac{(1-cos\theta)}{(1-cos\theta)}\left(\frac{sin\theta}{1+cos\theta}\right)} = \frac{1-cos\theta}{sin\theta}$$

$\frac{(1-cos\theta)sin\theta}{1-cos^2\theta}$

$\frac{(1-cos\theta)\cancel{sin\theta}}{sin^{\cancel{2}}\theta}$

$\frac{1-cos\theta}{sin\theta}$

E. COMMON DENOMINATOR

- Add two rational expressions with binomial (2 terms) denominators:

$$\frac{1+a}{1+a}\cdot\frac{1}{1-a}+\frac{1}{1+a}\cdot\frac{1-a}{1-a}=\frac{1+a}{(1-a)(1+a)}+\frac{1-a}{(1-a)(1+a)}=\frac{1+a+1-a}{(1-a)(1+a)}=\frac{2}{1-a^2}$$

EX. 6: Use the technique you learned above to prove the following identity:

$$\frac{1-\cos\theta}{1-\cos\theta}\cdot\frac{1}{1+\cos\theta}+\frac{1}{1-\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}=2\csc^2\theta$$

$$\frac{1-\cos\theta}{(1+\cos\theta)(1-\cos\theta)}+\frac{1+\cos\theta}{(1+\cos\theta)(1-\cos\theta)}$$

$$\frac{2}{(1+\cos\theta)(1-\cos\theta)}$$

$$\frac{2}{1-\cos^2\theta}$$

$$\frac{2}{\sin^2\theta}$$

$$2\left(\frac{1}{\sin^2\theta}\right)$$

$$\frac{2}{\sin^2\theta}$$

PROVE AN IDENTITY USING DOUBLE-ANGLE IDENTITIES

$$\boldsymbol{\sin 2\theta = 2\sin\theta\cos\theta} \qquad \boldsymbol{\cos 2\theta = \cos^2\theta - \sin^2\theta} \qquad \boldsymbol{\tan\theta = \frac{\sin 2\theta}{\cos 2\theta}}$$

EX. 7: Prove for all permissible values

$$\cot x=\frac{\sin 2x}{1-\cos 2x}$$

$$\frac{\cos x}{\sin x}$$

$$\frac{2\sin x\cos x}{1-1(1-2\sin^2 x)}$$

$$\frac{2\sin x\cos x}{1-1+2\sin^2 x}$$

$$\frac{\cancel{2}\cancel{\sin x}\cos x}{\cancel{2}\sin^{\cancel{2}} x}$$

$$\frac{\cos x}{\sin x}$$

PROVE MORE COMPLICATED IDENTITIES

EX. 8: Prove for all permissible values

$$\frac{1}{1+\sin x} = \frac{\sec x - \sin x \sec x}{\cos x}$$

$$\frac{\dfrac{1}{\cos x} - \sin x\left(\dfrac{1}{\cos x}\right)}{\cos x}$$

$$\frac{\dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x}}{\cos x}$$

$$\frac{\dfrac{1-\sin x}{\cos x}}{\cos x}$$

$$\frac{1-\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

$$\frac{1-\sin x}{\cos^2 x} \cdot \frac{(1+\sin x)}{(1+\sin x)}$$

$$\frac{1-\sin^2 x}{\cos^2 x(1+\sin x)}$$

$$\frac{\cancel{\cos^2 x}}{\cancel{\cos^2 x}(1+\sin x)}$$

$$\frac{1}{1+\sin x}$$

$$\frac{\sec x(1-\sin x)}{\cos x}$$

$$\frac{\dfrac{1}{\cos x}(1-\sin x)}{\cos x}$$

$$\frac{1-\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

ASSIGNMENT: 1) Worksheet 6.3: Proving Identities
2) pg. 314 # 1-8, 10-13, 15

PC 12 **WORKSHEET 6.3: PROVING IDENTITIES**

I. Show all steps when completing each of the following proofs

➔ Divide your paper into 4. Do one proof in each quarter page.

1. $sin\theta(sec\theta - csc\theta) = tan\theta - 1$
2. $sec^2\theta - sec^2\theta\, sin^2\theta = 1$
3. $sin^4\theta - cos^4\theta = 2\, sin^2\theta - 1$
4. $tan^2\theta(1 + cot^2\theta) = sec^2\theta$
5. $cot\theta = \frac{1+cot\theta}{1+tan\theta}$
6. $\frac{1+sec\theta}{sec\theta-1} = \frac{1+cos\theta}{1-cos\theta}$
7. $\frac{1+sin\theta}{1-sin\theta} = \frac{csc\theta+1}{csc\theta-1}$
8. $\frac{sin\theta+cos\theta tan\theta}{cot\theta} = 2tan\theta sin\theta$
9. $\frac{sin\theta}{1+cos\theta} = \frac{1-cos\theta}{sin\theta}$
10. $\frac{tan\theta}{sec\theta+1} = \frac{sec\theta-1}{tan\theta}$
11. $\frac{sin\theta}{1+cos\theta} + \frac{sin\theta}{1-cos\theta} = 2csc\theta$
12. $\frac{2}{1-sin\theta} + \frac{2}{1+sin\theta} = 4sec^2\theta$
13. $\frac{1}{1-sin\theta} = \frac{1+sin\theta}{cos^2\theta}$
14. $\frac{1-cos\theta}{sin\theta} = \frac{tan\theta-sin\theta}{tan\theta sin\theta}$
15. $1 + sec^2\theta\, sin^2\theta = sec^2\theta$
16. $1 + cot^2\theta = \frac{1}{sin^2\theta}$
17. $\frac{sin\theta}{1-sin^2\theta} = sec\theta\, tan\theta$
18. $\frac{cot\theta}{sec\theta} = \frac{1}{sin\theta} - sin\theta$
19. $\frac{tan^2\theta-sec^2\theta}{tan\theta-sec\theta} = tan\theta + sec\theta$
20. $\frac{cos\theta}{1-sin\theta} = sec\theta + tan\theta$
21. $\frac{1+tan\theta}{1+cot\theta} = \frac{sin\theta}{cos\theta}$
22. $tan^2\theta = sin^2\theta(1 + tan^2\theta)$
23. $sin^4\theta - cos^4\theta = 2\, sin^2\theta - 1$
24. $tan\theta + cot\theta = sec\theta csc\theta$
25. $\frac{2tan\theta}{sin^2\theta+cos^2\theta+tan^2\theta} = sin2\theta$
26. $(sin^2\theta - cos^2\theta)^2 - sin^2 2\theta = cos4\theta$
27. $\frac{cos2\theta+1}{sin2\theta} = \frac{cos\theta}{sin\theta}$
28. $\frac{sin2\theta}{1-cos2\theta} = 2csc2x - tanx$
29. $tan\theta = \frac{1-cos2\theta}{sin2\theta}$
30. $2cos\theta csc2\theta = csc\theta$