PC 12 SEC 6.2: SUM, DIFFERENCE, & DOUBLE-ANGLE IDENTITIES

INVESTIGATE:

- Determine whether the following "equations" are true. Verify when a = 1 and b = 2.

a) $3(a+b) = 3a + 3b$

$3(1+2) = 3(1) + 3(2)$ TRUE

$3(3) = 3 + 6$

$9 = 9$

b) $(a+b)^2 = a^2 + b^2$

$(1+2)^2 = (1)^2 + (2)^2$

$(3)^2 = 1 + 4$ FALSE

$9 \neq 5$

c) $\log(a+b) = \log a + \log b$

$\log(1+2) = \log(1) + \log(2)$

$\log 3 = \log 1 + \log 2$ FALSE

$0.477 \neq 0.301$

d) $\sin(a+b) = \sin a + \sin b$

$\sin(1+2) = \sin 1 + \sin 2$

$\sin 3 = \sin 1 + \sin 2$

$0.84 \neq 1.75$ FALSE

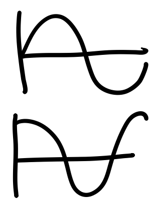
NOTE(s): 1) Very few functions can be "distributed" like we do in multiplication.
In Ch. 8 we will explore the many rules of logarithms
In 6.2 we will explore trigonometric sum & difference identities

2) To prove something is "always" true can be difficult.
To prove it is false is easy, just find a counterexample

INVESTIGATE EXPRESSIONS FOR $sin(\alpha + \beta)$ and $cos(\alpha + \beta)$

Sin

Cos



NOTE: This is not a PROOF

- Verify the given identity numerically by using $a = \frac{\pi}{2}$ & $b = \frac{\pi}{4}$

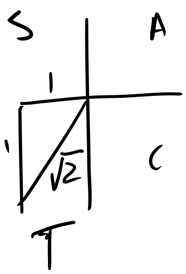
$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$\sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$

$= \sin\left(\frac{2\pi}{4} + \frac{\pi}{4}\right)$

$= \sin\left(\frac{3\pi}{4}\right)$

$= \frac{1}{\sqrt{2}}$



$\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)$

$= (1)\left(\frac{1}{\sqrt{2}}\right) + (0)\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{1}{\sqrt{2}}$

SUM IDENTITIES

$\sin(\alpha + \beta) = sin\alpha cos\beta + cos\alpha sin\beta$

$cos(\alpha + \beta) = cos\alpha cos\beta - sin\alpha sin\beta$

$\tan(\alpha + \beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha tan\beta}$

DIFFERENCE IDENTITIES

$\sin(\alpha - \beta) = sin\alpha cos\beta - cos\alpha sin\beta$

$cos(\alpha - \beta) = cos\alpha cos\beta + sin\alpha sin\beta$

$\tan(\alpha - \beta) = \frac{tan\alpha - tan\beta}{1 + tan\alpha tan\beta}$

EX. 1: Use the new identity to simplify each expression:

a) $sin55°cos35° + cos55°sin35°$ $\;$ ($\sin\alpha \sin\beta \quad \cos\alpha \sin\beta = \sin(\alpha+\beta)$)

$= \sin(55° + 35°)$

$= \sin(90°)$

$= 1$

b) $sin4\theta cos\theta + cos4\theta sin\theta$ $\;$ ($\alpha \; \beta \; \alpha \; \beta = \sin(\alpha+\beta)$)

$= \sin(4\theta + \theta)$

$= \sin(5\theta)$

EX. 2: Use the appropriate identity to simplify each expression:

a) $cos3\theta cos\theta + sin3\theta sin\theta$ $\;$ ($\alpha \; \beta \; \alpha \; \beta$) $= \cos(\alpha - \beta)$

$= \cos(3\theta - \theta)$

$= \cos(2\theta)$

b) $cos7xcos2x - sin7xsin2x$ $\;$ ($\alpha \; \beta \; \alpha \; \beta = \cos(\alpha+\beta)$)

$= \cos(7x + 2x)$

$= \cos(9x)$

c) $sin8xcos3x - cos8xsin3x$ $\;$ ($\alpha \; \beta \; \alpha \; \beta = \sin(\alpha - \beta)$)

$= \sin(8x - 3x)$

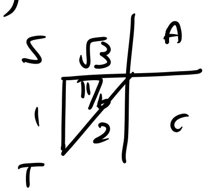
$= \sin(5x)$

d) $sin\frac{\pi}{6}cos\pi + cos\frac{\pi}{6}sin\pi$ $\;$ ($\alpha \; \beta \; \alpha \; \beta = \sin(\alpha+\beta)$)

$= \sin(\frac{\pi}{6} + \pi)$

$= \sin(\frac{7\pi}{6})$

$= -\frac{1}{2}$

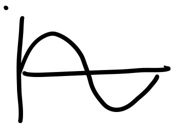


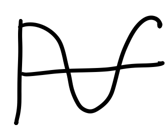
EX. 3: Use the appropriate identity to expand and simplify the following:

a) $\sin(\frac{\pi}{2} - \theta)$ $= \sin\frac{\pi}{2}\cos\theta - \cos\frac{\pi}{2}\sin\theta$

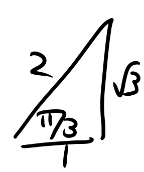
$= (1)\cos\theta - (0)\sin\theta$

$= \cos\theta$

Sin


Cos


b) $\cos(\pi - \frac{\pi}{3})$ $= \cos\pi\cos\frac{\pi}{3} + \sin\pi\sin\frac{\pi}{3}$



c) $\tan(\pi - \theta) = \dfrac{\sin(\pi-\theta)}{\cos(\pi-\theta)} = \dfrac{\sin\pi\cos\theta - \cos\pi\sin\theta}{\cos\pi\cos\theta + \sin\pi\sin\theta}$

$= \dfrac{(0)\cos\theta - (-1)\sin\theta}{(-1)\cos\theta + (0)\sin\theta}$

$= \dfrac{\sin\theta}{-\cos\theta}$

$= -\tan\theta$

EX. 4: Prove the following identities:

a) $$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$\cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta$

$(0)\cos\theta + (1)\sin\theta$

$\sin\theta \quad = \sin\theta$

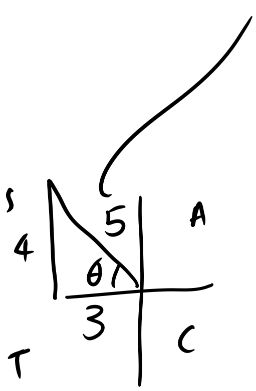
b) $$\sin(\pi - \theta) = \sin\theta$$

$\sin\pi\cos\theta - \cos\pi\sin\theta$

$(0)\cos\theta - (-1)\sin\theta$

$\sin\theta \quad = \sin\theta$

EX. 5: Given $\sin\theta = \frac{4}{5}$, where θ is in Quadrant II, evaluate the expression $\sin(\theta + \frac{\pi}{6})$

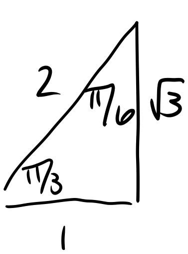


$$\sin\left(\theta + \frac{\pi}{6}\right) = \sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6}$$

$$= \left(\frac{4}{5}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{-3}{5}\right)\left(\frac{1}{2}\right)$$

$$= \frac{4\sqrt{3}}{10} - \frac{3}{10}$$

$$= \frac{4\sqrt{3} - 3}{10}$$



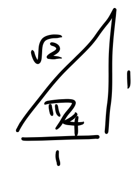
INVESTIGATE: DOUBLE ANGLE IDENTITIES (2θ):

a) Write the identity for $sin(\alpha + \beta)$. Let $\alpha = \theta$ and $\beta = \theta$ Simplify the expression.

$\sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$

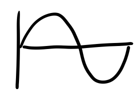
$\sin(2\theta) = 2\sin\theta\cos\theta$

b) Verify that the double angle identity for sine is true by making $\theta = \frac{\pi}{4}$



$\sin\left(2\left(\frac{\pi}{4}\right)\right) = 2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)$

$\sin\left(\frac{\pi}{2}\right) = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$



$1 = \frac{2}{\sqrt{4}} \rightarrow 1 = \frac{2}{2} \rightarrow 1 = 1$

c) Write the identity for $cos(\alpha + \beta)$. Let $\alpha = \theta$ and $\beta = \theta$. Simplify the expression.

$\cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$

$= \cos^2\theta - \sin^2\theta$

d) Use the Pythagorean identity $sin^2\theta + cos^2\theta = 1$ to prove the other two double angle identities for cosine

i) $cos2\theta = 2cos^2\theta - 1$

$\cos^2\theta - \sin^2\theta$	
$\cos^2\theta - (1-\cos^2\theta)$	
$\cos^2\theta - 1 + \cos^2\theta$	
$2\cos^2\theta - 1$	$= 2\cos^2\theta - 1$

ii) $cos2\theta = 1 - 2sin^2\theta$

$\cos^2\theta - \sin^2\theta$	
$(1-\sin^2\theta) - \sin^2\theta$	
$1 - \sin^2\theta - \sin^2\theta$	
$1 - 2\sin^2\theta =$	$1 - 2\sin^2\theta$

DOUBLE ANGLE IDENTITIES

$\boldsymbol{sin2\theta = 2sin\theta cos\theta}$

$\boldsymbol{cos2\theta = cos^2\theta - sin^2\theta}$

$\boldsymbol{cos2\theta = 2cos^2\theta - 1}$

$\boldsymbol{cos2\theta = 1 - 2sin^2\theta}$

$\boldsymbol{tan\theta = \frac{sin2\theta}{cos2\theta}}$

$\boldsymbol{tan\theta = \frac{2tan\theta}{1 - tan^2\theta}}$

COMPARING:

$\sin^2 \vartheta$, $\sin 2\vartheta$, $2 \sin \vartheta$

EX. 6: Use algebra to simplify:

a) $\sin(\vartheta + \vartheta) =$ $\sin 2\theta$

b) $\sin \vartheta + \sin \vartheta =$ $2\sin\theta$

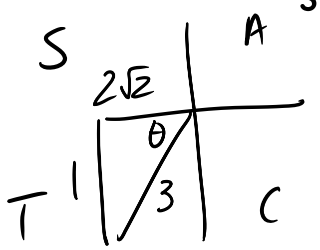
c) $\sin \vartheta \sin \vartheta =$ $\sin^2\theta$

d) $\cos^2 \vartheta \cos \vartheta =$ $\cos^3\theta$

e) $3 \sin \vartheta \sin \vartheta =$ $3\sin^2\theta$

f) $3 \sin \vartheta + \sin \vartheta =$ $4\sin\theta$

EX. 7: Given $\sin \vartheta = -\frac{1}{3}$, where ϑ is in Quadrant III, evaluate:



$3^2 - 1^2 = 8$

$\sqrt{8} = \sqrt{2 \cdot 2 \cdot 2} = 2\sqrt{2}$

Rather than memorize the identity for **$\tan 2\vartheta$**, just replace it with $\frac{\sin 2\vartheta}{\cos 2\vartheta}$

a) $\sin 2\vartheta =$

$2\sin\theta\cos\theta$

$2\left(-\frac{1}{3}\right)\left(-\frac{2\sqrt{2}}{3}\right)$

$\frac{4\sqrt{2}}{9}$

b) $\cos 2\vartheta =$

$\cos^2\theta - \sin^2\theta$

$= \left(\frac{-2\sqrt{2}}{3}\right)^2 - \left(\frac{-1}{3}\right)^2$

$= \frac{8}{9} - \frac{1}{9}$

$= \frac{7}{9}$

$\tan 2\vartheta = \frac{\sin 2\theta}{\cos 2\theta}$

$= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$

$= \frac{\frac{4\sqrt{2}}{\cancel{9}}}{\frac{7}{\cancel{9}}}$

$= \frac{4\sqrt{2}}{7}$

EX. 8: Use the double angle identities to write each expression as a single trigonometric function.

$2\sin\theta\cos\theta = \sin 2\theta$

a) $2\sin\frac{\pi}{3}\cos\frac{\pi}{3} = \sin\left(\frac{2\pi}{3}\right)$

$\cos^2\theta - \sin^2\theta = \cos 2\theta$

b) $\cos^2\frac{\pi}{4} - \sin^2\frac{\pi}{4} = \cos\left(\frac{2\pi}{4}\right)$

$\cos\left(\frac{\pi}{2}\right)$

$1 - 2\sin^2\boxed{\theta} = \cos\boxed{2\theta}$

c) $1 - 2\sin^2\boxed{5} = \cos\boxed{10}$

$2\cos^2\boxed{\theta} - 1 = \cos 2\theta$

d) $2\cos^2\boxed{\beta} - 1 = \cos 2\beta$

$\frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}} = \frac{\sin 2\theta}{2}$ OR

e) $\sin\vartheta\cos\vartheta = \frac{1}{2}\sin 2\theta$

$3(2\cos^2\theta - 1) = 3\cos 2\theta$

f) $6\cos^2\vartheta - 3 =$

g) $7 - 14\sin^2\vartheta =$

$= 7(1 - 2\sin^2\theta)$

$= 7\cos 2\theta$

h) $\sin 7\cos 7 = \frac{1}{2}\sin 14$ or $\frac{\sin 14}{2}$

i) $\frac{\sin 2\vartheta}{2\sin\vartheta} = \frac{\cancel{2\sin\theta}\cos\theta}{\cancel{2\sin\theta}}$

$= \cos\theta$

j) $2\csc 2\vartheta\cos\vartheta = 2\left(\frac{1}{\sin 2\theta}\right)\cos\theta$

$= \frac{2\cos\theta}{\sin 2\theta}$

$= \frac{\cancel{2\cos\theta}}{\cancel{2}\sin\theta\cancel{\cos\theta}}$

$= \frac{1}{\sin\theta}$

$= \csc\theta$

EX. 9: Prove the following identities (algebraically)

a) $$\frac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$$

$$\frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$$

$$\frac{1-1+2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$\frac{\cancel{2}\sin^{\cancel{2}}\theta}{\cancel{2}\,\cancel{\sin\theta}\cos\theta}$$

$$\frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\cos\theta}$$

b) $$\frac{(\sin\theta+\cos\theta)^2}{\sin 2\theta} = \boxed{\csc 2\theta}+1$$

$$\frac{(\sin\theta+\cos\theta)(\sin\theta+\cos\theta)}{\sin 2\theta} \qquad \frac{1}{\sin 2\theta} + \frac{1}{1}\cdot\frac{\sin 2\theta}{\sin 2\theta}$$

$$\frac{\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta}{\sin 2\theta} \qquad \frac{1}{\sin 2\theta}+\frac{\sin 2\theta}{\sin 2\theta}$$

$$\frac{(\sin^2\theta+\cos^2\theta)+2\sin\theta\cos\theta}{\sin 2\theta} \qquad \frac{1+\boxed{\sin 2\theta}}{\sin 2\theta}$$

$$\frac{1+2\sin\theta\cos\theta}{\sin 2\theta} = \frac{1+\boxed{2\sin\theta\cos\theta}}{\sin 2\theta}$$

c) $$\cos 2\theta = \frac{\csc^2\theta-2}{\csc^2\theta}$$

$$1-2\sin^2\theta \qquad \frac{\frac{1}{\sin^2\theta}-2}{\frac{1}{\sin^2\theta}}$$

$$\frac{\frac{1}{\sin^2\theta}-\frac{2\sin^2\theta}{\sin^2\theta}}{\frac{1}{\sin^2\theta}}$$

$$\frac{\frac{1-2\sin^2\theta}{\cancel{\sin^2\theta}}}{\frac{1}{\cancel{\sin^2\theta}}}$$

$$1-2\sin^2\theta = 1-2\sin^2\theta$$

d) $$\frac{\sin 2x}{2\sin^2 x} = \cot x$$

$$\frac{\cancel{2}\cancel{\sin x}\cos x}{\cancel{2}\sin^{\cancel{2}} x}$$

$$\frac{\cos x}{\sin x} = \frac{\cos x}{\sin x}$$

e) $$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\frac{1+(2\cos^2 x - 1)}{2}$$

$$\frac{\cancel{2}\cos^2 x}{\cancel{2}}$$

$$\cos^2 x = \cos^2 x$$

SIMPLIFY EXPRESSIONS USING SUM, DIFFERENCE, & DOUBLE-ANGLE IDENTITIES

EX. 10: Write each expression as a single trigonometric function. $2\sin\theta\cos\theta = \sin 2\theta$

a) $\cos 88° \cos 35° + \sin 88° \sin 35°$

$= \cos(88° - 35°)$

$= \cos(53°)$

b) $2\sin\frac{\pi}{12}\cos\frac{\pi}{12}$

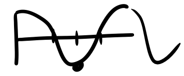
$= \sin\left(2\left(\frac{\pi}{12}\right)\right)$

$= \sin\left(\frac{\pi}{6}\right)$

SIMPLIFY EXPRESSIONS USING IDENTITIES

EX. 11: Consider the expression $\frac{\sin 2x}{\cos 2x+1}$ $\quad P = \frac{2\pi}{2} = \pi$

a) What are the non-permissible values for the expression



$\cos 2x + 1 \neq 0$ $\quad$ let $a = 2x$

$\cos 2x \neq -1$ $\quad \cos a \neq -1$ $\quad x \neq \frac{\pi}{2}$

$a \neq \pi$

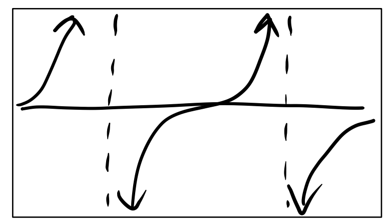
$2x \neq \pi$ $\quad x \neq \frac{\pi}{2} + \pi n,\ n \in I$

b) Simplify the expression to one of the three primary trigonometric functions.

$$\frac{\sin 2x}{(\cos 2x)+1} = \frac{2\sin x\cos x}{(2\cos^2 x - 1) + 1} = \frac{\cancel{2}\sin x\cancel{\cos x}}{\cancel{2}\cos^{\cancel{2}} x} = \frac{\sin x}{\cos x} = \tan x$$

c) Verify your answer graphically over the interval $[0, 2\pi)$

same graph



DETERMINE EXACT TRIGONOMETRIC VALUES FOR ANGLES

EX. 12: Determine the exact value for each expression

a) $\cos 165° = \cos(45° + 120°) = \cos 45° \cos 120° - \sin 45° \sin 120°$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{-1-\sqrt{3}}{2}$$

b) $\tan \frac{11\pi}{12}$

Method 1: Using Difference Identity

$$\tan\left(\frac{2\pi}{12} + \frac{9\pi}{12}\right)$$

$$= \tan\left(\frac{\pi}{6} + \frac{3\pi}{4}\right) = \frac{\tan \pi/6 + \tan 3\pi/4}{1 - \tan \pi/6 \tan 3\pi/4}$$

$= \ldots$

Method 2: Using Quotient Identity

$$\frac{\sin \frac{11\pi}{12}}{\cos \frac{11\pi}{12}} = \frac{\sin\left(\frac{\pi}{6} + \frac{3\pi}{4}\right)}{\cos\left(\frac{\pi}{6} + \frac{3\pi}{4}\right)} = \ldots$$

ASSIGNMENT: 1) Worksheet 6.2 Trigonometri
2) pg. 306 # 1-9, 12, 15, 17, 20

PC 12 **WORKSHEET 6.2: TRIGONOMETRIC IDENTITIES**

1. Prove

a) $\cos(\pi + \theta) = -cos\theta$

c) $\cos\left(\frac{3\pi}{2} + \theta\right) = sin\theta$

b) $\sin\left(\frac{\pi}{6} + x\right) + \sin\left(\frac{\pi}{6} - x\right) = cosx$

d) $\sin(x + y) + \sin(x - y) = 2sinxcosy$

2. $cos\theta = -\frac{2}{3}$, where θ is in Quadrant II, evaluate the expression $\sin(\theta + \frac{\pi}{3})$

COMPARING:

$\sin^2 \vartheta + \cos^2 \vartheta = 1$

$\cos 2\vartheta = \cos^2 \vartheta - \sin^2 \vartheta$

3. Simplify the following (be careful)

a) $\sin^2 \vartheta + \cos^2 \vartheta =$

b) $\cos^2 \vartheta + \sin^2 \vartheta =$

c) $\cos^2 \vartheta - \sin^2 \vartheta =$

d) $\sin^2 \vartheta - \cos^2 \vartheta =$

e) $-\sin^2 \vartheta - \cos^2 \vartheta =$

f) $1 - \cos^2 \vartheta =$

g) $1 - \sin^2 \vartheta =$

h) $\cos^2 \vartheta - 1 =$

i) $\sin^2 \vartheta - 1 =$

j) $2\cos^2 \vartheta - 1 =$

k) $2\sin^2 \vartheta - 1 =$

l) $1 - 2\sin^2 \vartheta =$

m) $1 - 2\cos^2 \vartheta =$

n) $2 - 2\sin^2 \vartheta =$

o) $2\cos^2 \vartheta - 2 =$

p) $4\cos^2 \vartheta - 4 =$

q) $4\cos^2 \vartheta - 2 =$

r) $6\cos^2 \vartheta - 3 =$

Work with angle or the number in front of the identity

4. Simplify the following (be careful)

a) $\sin^2 \beta + \cos^2 \beta =$

b) $\sin^2 2x + \cos^2 2x =$

c) $\sin^2 3x + \cos^2 3x =$

d) $\cos^2 \gamma - \sin^2 \gamma =$

e) $\cos^2 3x - \sin^2 3x =$

f) $2 \sin 5x \cos 5x =$

g) $1 - 2\sin^2 6x =$

h) $10 \sin \vartheta \cos \vartheta$

i) $6 \sin 2x \cos 2x =$

j) $10 \cos^2 7x - 5 =$